

Bäcklund Transformations and the Atiyah-Ward Ansatz for Noncommutative Anti-Self-Dual Yang-Mills Equations

Claire R. Gilson[†], Masashi Hamanaka[‡] and Jonathan J. C. Nimmo[†]

[†]*Department of Mathematics, University of Glasgow
Glasgow G12 8QW, UK*

[‡]*Department of Mathematics, University of Nagoya,
Nagoya, 464-8602, JAPAN*

Abstract

We present Bäcklund transformations for the noncommutative anti-self-dual Yang-Mills equation where the gauge group is $G = GL(2)$ and use it to generate a series of exact solutions from a simple seed solution. The solutions generated by this approach are represented in terms of quasideterminants. We also explain the origins of all of the ingredients of the Bäcklund transformations within the framework of noncommutative twistor theory. In particular we show that the generated solutions belong to a noncommutative version of the Atiyah-Ward ansatz.

Key Words

noncommutative integrable systems, anti-self-dual Yang-Mills equation, quasideterminant solutions, Atiyah-Ward ansatz, Penrose-Ward transformation

1. Introduction

In both mathematics and physics, a noncommutative extension is a natural generalization of a commutative theory that sometimes leads to a new and deeper understanding of that theory. While matrix (or more general, non-Abelian) generalizations have been studied for a long time, the generalization to noncommutative flat spaces, triggered by developments in string theory (see e.g. Douglas and Nekrasov (2002); Szabo (2003)), has become a hot topic recently. These generalizations are realized by replacing all products in the commutative theory with associative but noncommutative Moyal products. In gauge theories, such a noncommutative extension is equivalent to the presence of a background magnetic field. Many successful applications of the analysis of noncommutative solitons to D -brane dynamics have been made. (Here we use the word “soliton” as a stable configuration which possesses localized energy densities and hence includes static configurations such as instantons.)

Integrable systems and soliton theory can also be extended to a noncommutative setting and yield interesting results and applications. (For reviews, see e.g. Dimakis and Müller-Hoissen (2004); Hamanaka (2003); Hamanaka (2005a); Hamanaka and Toda (2003); Kupershmidt (2000); Lechtenfeld (2008); Mazzanti (2007); Tamassia (2005).) Among them, the noncommutative anti-self-dual Yang-Mills (ASDYM) equation in 4-dimensions is important because in the Euclidean signature $(++++)$, the ADHM construction can be used to find all exact instanton solutions and gives rise to new physical objects such as $U(1)$ instantons (Nekrasov and Schwarz (1998)). In the split signature $(++--)$, many noncommutative integrable equations can be derived from the noncommutative ASDYM equation by a reduction process (See Hamanaka (2005b); Hamanaka (2006) and references therein). Integrable aspects of the noncommutative ASDYM equation can be understood in the geometrical framework of noncommutative twistor theory (Brain (2005); Brain and Majid (2008); Hannabuss (2001); Horváth, Lechtenfeld and Wolf (2002); Ihl and Uhlmann (2003); Kapustin, Kuznetsov and Orlov (2001); Lechtenfeld and Popov (2002); Takasaki (2001)). Therefore it is worth studying the integrable aspects of the noncommutative ASDYM equation in detail both for the applications to lower-dimensional integrable equations and to the corresponding physical situations in the framework of a noncommutative analogue of $N = 2$ string theory (Lechtenfeld, Popov and Spindig (2001a); Lechtenfeld, Popov and Spindig (2001b)). Here solitons are not, in general, static and suggest the existence of some kinds of new configurations. For these purposes, Bäcklund transformations play an important role in constructing exact solutions and revealing an (infinite-dimensional) symmetry of the solution space in terms of the transformation group. Also, a twistor description is useful for a discussion of the origin of the transformations and for checking whether or not the group action is transitive.

In the present paper, we give Bäcklund transformations for the noncommutative ASDYM equation where the gauge group is $G = GL(2)$ and use them to generate a se-

quence of exact solutions from a simple seed solution. This approach gives both finite action solutions (instantons) and infinite action solutions (such as nonlinear plane waves). The solutions obtained are written in terms of quasideterminants (Gelfand and Retakh (1991); Gelfand and Retakh (1992)) which appear also in the construction of exact soliton solutions in lower-dimensional noncommutative integrable equations such as the Toda equation (Etingof, Gelfand and Retakh (1997); Etingof, Gelfand and Retakh (1998); Li and Nimmo (2008); Li and Nimmo (2009)), the KP and KdV equations (Dimakis and Müller-Hoissen (2007); Etingof, Gelfand and Retakh (1997); Gilson and Nimmo (2007); Hamanaka (2007)), the Hirota-Miwa equation (Gilson, Nimmo and Ohta (2007); Li, Nimmo and Tamizhmani (2009); Nimmo (2006)), the mKP equation (Gilson, Nimmo and Sooman (2008a); Gilson, Nimmo and Sooman (2008b)), the Schrödinger equation (Goncharenko and Veselov (1998); Samsonov and Pecheritsin (2004)), the Davey-Stewartson equation (Gilson and Macfarlane (2009)), the dispersionless equation (Hassan (2009)), and the chiral model (Haider and Hassan (2008)), where they play the role that determinants do in the corresponding commutative integrable systems. We also clarify the origin of the results from the viewpoint of noncommutative twistor theory by using noncommutative Penrose-Ward correspondence or by solving a noncommutative Riemann-Hilbert problem. It is shown that the solutions generated belong to a noncommutative version of the *Atiyah-Ward ansatz* (Atiyah and Ward (1977)).

The discussion and strategy used in this paper are simple noncommutative generalizations of those used in the commutative case (Corrigan, Fairlie, Yates and Goddard (1978); Mason, Chakravarty and Newman (1988); Mason and Woodhouse (1996)). In the commutative limit, our results coincide in part with the known results but in the noncommutative case, there are several nontrivial points. Firstly, in Section 3.2 we show that quasideterminants are ideally suited to the noncommutative extension of the known results and greatly simplify the proofs of the Bäcklund transformations even in the commutative limit. The simple quasideterminant representations of Yang's J -matrix are new and imply the important result that the Bäcklund transformation is not just a gauge transformation. It is possible for the noncommutative twistor description to work as it does in the commutative setting because one of the three local coordinates can be taken to be a commutative variable.

In our treatment, all dependent variables belong to a ring, which has an associative but not necessarily commutative product. Hence the results we obtain are available in any noncommutative settings such as the Moyal-deformed, matrix or quaternion-valued ASDYM equations.

2. The noncommutative ASDYM equation

Let us consider noncommutative Yang-Mills theories in 4-dimensions where the gauge group is $GL(N)$ and the real coordinates are x^μ , $\mu = 0, 1, 2, 3$. In the rest of the paper,

we follow the conventions of notation given in Mason and Woodhouse (1996).

(a) *The noncommutative ASDYM equation*

The noncommutative ASDYM equation is derived from the compatibility condition of the linear system

$$\begin{aligned} L\psi &:= (D_w - \zeta D_{\bar{z}})\psi = (\partial_w + A_w - \zeta(\partial_{\bar{z}} + A_{\bar{z}}))\psi = 0, \\ M\psi &:= (D_z - \zeta D_{\bar{w}})\psi = (\partial_z + A_z - \zeta(\partial_{\bar{w}} + A_{\bar{w}}))\psi = 0, \end{aligned} \quad (1)$$

where $A_z, A_w, A_{\bar{z}}, A_{\bar{w}}$ and $D_z, D_w, D_{\bar{z}}, D_{\bar{w}}$ denote gauge fields and covariant derivatives in Yang-Mills theory, respectively. The (commutative) variable ζ is a local coordinate of a one-dimensional complex projective space \mathbb{CP}_1 , and is called the *spectral parameter*. We note that ψ is not regular at $\zeta = \infty$ because if it were regular, by Liouville's theorem it would be a constant function and the gauge fields would be flat (see e.g. Mason and Woodhouse (1996)).

The four complex coordinates z, \bar{z}, w, \bar{w} are double null coordinates (Mason and Woodhouse (1996)). By imposing the corresponding reality conditions, we can realize real spaces with different signatures, that is,

- the Euclidean space, obtained by putting $\bar{w} = -\tilde{w}; \bar{z} = \tilde{z}$, for example,

$$\begin{bmatrix} \tilde{z} & w \\ \tilde{w} & z \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x^0 + ix^1 & -(x^2 - ix^3) \\ x^2 + ix^3 & x^0 - ix^1 \end{bmatrix}, \quad (2)$$

- the Ultrahyperbolic space, obtained by putting $\bar{w} = \tilde{w}; \bar{z} = \tilde{z}$, for example,

$$\begin{bmatrix} \tilde{z} & w \\ \tilde{w} & z \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{bmatrix} \quad \text{or} \quad z, w, \tilde{z}, \tilde{w} \in \mathbb{R}. \quad (3)$$

The compatibility condition $[L, M] = 0$, gives rise to a quadratic polynomial in ζ and each coefficient yields the noncommutative ASDYM equations, with explicit representations

$$\begin{aligned} F_{wz} &= \partial_w A_z - \partial_z A_w + [A_w, A_z] = 0, \\ F_{\tilde{w}\tilde{z}} &= \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + [A_{\tilde{w}}, A_{\tilde{z}}] = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} &= \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_{\tilde{w}} A_w - \partial_w A_{\tilde{w}} + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0, \end{aligned} \quad (4)$$

which is equivalent to the ASD condition for gauge fields $F_{\mu\nu} = - * F_{\mu\nu}$ in the real representation where the symbol $*$ is the Hodge dual operator.

When the compatibility conditions are satisfied, the linear system (1) has N independent solutions. Hence the solution $\psi(x; \zeta)$ can be interpreted as an $N \times N$ matrix whose columns are the N independent solutions.

Gauge transformations act on the linear system (1) as

$$L \mapsto g^{-1} L g, \quad M \mapsto g^{-1} M g, \quad \psi \mapsto g^{-1} \psi, \quad g \in G. \quad (5)$$

(b) *The noncommutative Yang equation and J, K -matrices*

Here we discuss the different potential forms of the noncommutative ASDYM equations such as the noncommutative J, K -matrix formalisms and the noncommutative Yang equation, which were already presented by e.g. Takasaki (2001).

Let us first discuss the *J -matrix formalism* of the noncommutative ASDYM equation (4). The first and second equations of (4) are the compatibility conditions of

$$D_z h = 0, \quad D_w h = 0, \quad \text{and} \quad D_{\tilde{z}} \tilde{h} = 0, \quad D_{\tilde{w}} \tilde{h} = 0, \quad (6)$$

respectively. Here h and \tilde{h} are $N \times N$ matrices, whose N columns of h and \tilde{h} are independent solutions of the linear systems. The existence of them is formally proved in the case of the Moyal deformation by Takasaki (2001) and presumed here. These equations can be satisfied by choosing

$$A_z = -(\partial_z h)h^{-1}, \quad A_w = -(\partial_w h)h^{-1}, \quad A_{\tilde{z}} = -(\partial_{\tilde{z}} \tilde{h})\tilde{h}^{-1}, \quad A_{\tilde{w}} = -(\partial_{\tilde{w}} \tilde{h})\tilde{h}^{-1}. \quad (7)$$

By defining $J = \tilde{h}^{-1}h$, the third equation of (4) becomes

$$\partial_z (J^{-1} \partial_{\tilde{z}} J) - \partial_w (J^{-1} \partial_{\tilde{w}} J) = 0. \quad (8)$$

This equation is called the *noncommutative Yang equation* and the matrix J is called *Yang's J -matrix*.

Gauge transformations act on h and \tilde{h} as

$$h \mapsto g^{-1}h, \quad \tilde{h} \mapsto g^{-1}\tilde{h}, \quad g \in G. \quad (9)$$

Hence Yang's J -matrix is gauge invariant. Gauge fields are obtained from a solution J of the noncommutative Yang's equation via a decomposition $J = \tilde{h}^{-1}h$, and (7). The different decompositions correspond to different choices of gauge.

There is another potential form of the noncommutative ASDYM equation, known as the *K -matrix formalism*. In the gauge in which $A_w = A_z = 0$, the third equation of (4) becomes $\partial_z A_{\tilde{z}} - \partial_w A_{\tilde{w}} = 0$. This implies the existence of a potential K such that $A_{\tilde{z}} = \partial_w K, A_{\tilde{w}} = \partial_z K$. Then the second equation of (4) becomes

$$\partial_z \partial_{\tilde{z}} K - \partial_w \partial_{\tilde{w}} K + [\partial_w K, \partial_z K] = 0. \quad (10)$$

This gauge is suitable for the discussion of (binary) Darboux transformations for (noncommutative) ASDYM equation (Gilson, Nimmo and Ohta (1998); Nimmo, Gilson and Ohta (2000); Saleem, Hassan and Siddiq (2007)).

3. Bäcklund transformation for the noncommutative ASDYM equation

In this section, we present two kind of Bäcklund transformations which leave the noncommutative Yang equation for $G = GL(2)$ invariant. This is a noncommutative version

of the Corrigan-Fairlie-Yates-Goddard transformation (Corrigan, Fairlie, Yates and Goddard (1978)). This transformation generates a class of exact solutions which belong to a noncommutative version of the *Atiyah-Ward ansatz* (Atiyah and Ward (1978)) labeled by a nonnegative integer $l \in \mathbb{Z}_{\geq 0}$. The origin of these results will be clarified in the next section.

In order to discuss Bäcklund transformations for the noncommutative Yang equation, we parameterize the 2×2 matrix J as

$$J = \begin{bmatrix} f - gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{bmatrix}. \quad (11)$$

This parameterization is always possible when f and b are invertible. In contrast with the commutative case, where only f appears, in the noncommutative setting, we need to introduce another variable b . In the commutative limit we may choose $b = f$.

Then the noncommutative Yang equation (8) is decomposed as

$$\begin{aligned} \partial_z(f^{-1}g_{\bar{z}}b^{-1}) - \partial_w(f^{-1}g_{\bar{w}}b^{-1}) &= 0, & \partial_{\bar{z}}(b^{-1}e_zf^{-1}) - \partial_{\bar{w}}(b^{-1}e_wf^{-1}) &= 0, \\ \partial_z(b_{\bar{z}}b^{-1}) - \partial_w(b_{\bar{w}}b^{-1}) - e_zf^{-1}g_{\bar{z}}b^{-1} + e_wf^{-1}g_{\bar{w}}b^{-1} &= 0, \\ \partial_z(f^{-1}f_{\bar{z}}) - \partial_w(f^{-1}f_{\bar{w}}) - f^{-1}g_{\bar{z}}b^{-1}e_z + f^{-1}g_{\bar{w}}b^{-1}e_w &= 0, \end{aligned} \quad (12)$$

where subscripts denote partial derivatives.

(a) *The noncommutative Corrigan-Fairlie-Yates-Goddard transformation*

The noncommutative Corrigan-Fairlie-Yates-Goddard transformation is a composition of the following two Bäcklund transformations for the noncommutative Yang equations (12).

- β -transformation (Mason and Woodhouse (1996)):

$$\begin{aligned} e_w^{\text{new}} &= -f^{-1}g_{\bar{z}}b^{-1}, e_z^{\text{new}} = -f^{-1}g_{\bar{w}}b^{-1}, g_{\bar{z}}^{\text{new}} = -b^{-1}e_wf^{-1}, g_{\bar{w}}^{\text{new}} = -b^{-1}e_zf^{-1}, \\ f^{\text{new}} &= b^{-1}, b^{\text{new}} = f^{-1}. \end{aligned} \quad (13)$$

The first four equations can be interpreted as integrability conditions for the first two equations in (12). We can easily check that the last two equations in (12) are invariant under this transformation.

- γ_0 -transformation (Gilson, Hamanaka and Nimmo (2009)):

$$\begin{bmatrix} f^{\text{new}} & g^{\text{new}} \\ e^{\text{new}} & b^{\text{new}} \end{bmatrix} = \begin{bmatrix} b & e \\ g & f \end{bmatrix}^{-1} = \begin{bmatrix} (b - ef^{-1}g)^{-1} & (g - fe^{-1}b)^{-1} \\ (e - bg^{-1}f)^{-1} & (f - gb^{-1}e)^{-1} \end{bmatrix}. \quad (14)$$

This follows from the fact that the transformation $\gamma_0 : J \mapsto J^{\text{new}}$ is equivalent to the simple conjugation $J^{\text{new}} = C_0^{-1}JC_0$, $C_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which clearly leaves the noncommutative Yang equation (8) invariant. The relation (14) is derived by comparing elements in this transformation.

It is easy to see that $\beta \circ \beta = \gamma_0 \circ \gamma_0 = id$, the identity transformation.

(b) *Exact noncommutative Atiyah-Ward ansatz solutions*

Now we construct exact solutions by using a chain of Bäcklund transformations from a seed solution. Let us consider $b = e = f = g = \Delta_0^{-1}$. We can easily find that the decomposed noncommutative Yang equation is reduced to a noncommutative linear equation $(\partial_z \partial_{\tilde{z}} - \partial_w \partial_{\tilde{w}}) \Delta_0 = 0$. (We note that for the Euclidean space, this is the noncommutative Laplace equation because of the reality condition $\tilde{w} = -\tilde{w}$.) Hence we can generate two series of exact solutions R_l and R'_l by iterating the β - and γ_0 -transformations one after the other as follows:

$$\begin{array}{ccccccc} R_0 & \xrightarrow{\alpha} & R_1 & \xrightarrow{\alpha} & R_2 & \xrightarrow{\alpha} & R_3 & \xrightarrow{\alpha} & R_4 & \longrightarrow & \cdots \\ & \searrow \beta & \uparrow \gamma_0 & \searrow \beta & \uparrow \gamma_0 & \searrow \beta & \uparrow \gamma_0 & \searrow \beta & \uparrow \gamma_0 & & \\ & & R'_1 & \xrightarrow{\alpha'} & R'_2 & \xrightarrow{\alpha'} & R'_3 & \xrightarrow{\alpha'} & R'_4 & \longrightarrow & \cdots \end{array}$$

where $\alpha = \gamma_0 \circ \beta : R_l \rightarrow R_{l+1}$ and $\alpha' = \beta \circ \gamma_0 : R'_l \rightarrow R'_{l+1}$. These two kind of series of solutions in fact arise from some class of noncommutative Atiyah-Ward ansatz. The explicit form of the solutions R_l or R'_l can be represented in terms of quasideterminants whose elements Δ_i ($i = -l+1, -l+2, \dots, l-1$) satisfy

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}, \quad -l+1 \leq i \leq l-2 \quad (l \geq 2), \quad (15)$$

which imply that every element Δ_i is a solution of the noncommutative linear equation $(\partial_z \partial_{\tilde{z}} - \partial_w \partial_{\tilde{w}}) \Delta_i = 0$. A brief introduction of quasideterminants is given in Appendix A.

The results are as follows:

- noncommutative Atiyah-Ward ansatz solutions R_l

The elements in J_l are given explicitly in terms of quasideterminants of the same $(l+1) \times (l+1)$ matrix:

$$\begin{aligned} b_l &= \left[\begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \boxed{\Delta_0} \end{array} \right]^{-1}, & f_l &= \left[\begin{array}{cccc} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{array} \right]^{-1}, \\ e_l &= \left[\begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \boxed{\Delta_{-l}} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{array} \right]^{-1}, & g_l &= \left[\begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\Delta_l} & \Delta_{l-1} & \cdots & \Delta_0 \end{array} \right]^{-1}. \end{aligned}$$

In the commutative limit, we can easily check by using (50) that $b_l = f_l$. The ansatz R_0 leads again to the so called the *Corrigan-Fairlie-'t Hooft-Wilczek* ansatz (Corrigan and Fairlie (1977); 't Hooft (1976), Wilczek (1977)).

- noncommutative Atiyah-Ward ansatz solutions R'_l

The elements in J'_l are given explicitly in terms of quasideterminants of the $l \times l$ matrices:

$$b'_l = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \end{vmatrix}, \quad f'_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \boxed{\Delta_0} \end{vmatrix},$$

$$e'_l = \begin{vmatrix} \Delta_{-1} & \Delta_{-2} & \cdots & \boxed{\Delta_{-l}} \\ \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_{-1} \end{vmatrix}, \quad g'_l = \begin{vmatrix} \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \Delta_2 & \Delta_1 & \cdots & \Delta_{3-l} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\Delta_l} & \Delta_{l-1} & \cdots & \Delta_1 \end{vmatrix}.$$

In the commutative case, $b'_l = f'_l$ also holds. For $l = 1$, we get $b'_1 = f'_1 = \Delta_0$, $e'_1 = \Delta_{-1}$, $g'_1 = \Delta_1$ and then the relation (15) implies that $e'_{1,z} = f'_{1,\bar{w}}$, $e'_{1,w} = f'_{1,\bar{z}}$, $b'_{1,z} = g'_{1,\bar{w}}$, $b'_{1,w} = g'_{1,\bar{z}}$, and leads to the Corrigan-Fairlie-'t Hooft-Wilczek ansatz as first pointed out by Yang (1977).

The γ_0 -transformation is proved simply using the noncommutative Jacobi identity (53) applied to the four corner elements. For example,

$$b_l^{-1} = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \ddots & \vdots \\ \Delta_{l-1} & \cdots & \boxed{\Delta_0} \end{vmatrix} - \begin{vmatrix} \Delta_1 & \cdots & \Delta_{2-l} \\ \vdots & \ddots & \vdots \\ \boxed{\Delta_l} & \cdots & \Delta_1 \end{vmatrix} \begin{vmatrix} \boxed{\Delta_0} & \cdots & \Delta_{1-l} \\ \vdots & \ddots & \vdots \\ \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1} \begin{vmatrix} \Delta_{-1} & \cdots & \boxed{\Delta_{-l}} \\ \vdots & \ddots & \vdots \\ \Delta_{l-2} & \cdots & \Delta_{-1} \end{vmatrix}$$

$$= (f'_l - g'_l b_l^{-1} e'_l).$$

The proof of the β -transformation uses both the noncommutative Jacobi identity (53) and also the homological relations (54). We will consider the first equation in the β -transformation:

$$e'_{l,w} = f_{l-1}^{-1} g_{l-1,z} b_{l-1}^{-1}. \quad (16)$$

The RHS is equal to

$$- b'_l g_{l-1} (g_{l-1}^{-1})_z g_{l-1} f'_l. \quad (17)$$

In this, it follows from (54) that the first two and last two factors are

$$b'_l g_{l-1} = \begin{vmatrix} \boxed{0} & \Delta_{-1} & \cdots & \Delta_{1-l} \\ 0 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & & \vdots \\ 0 & \Delta_{l-3} & \cdots & \Delta_{-1} \\ 1 & \Delta_{l-2} & \cdots & \Delta_0 \end{vmatrix}, \quad g_{l-1} f'_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & \vdots & & \vdots & \vdots \\ \Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_0 & \Delta_{-1} \\ 1 & 0 & \cdots & 0 & \boxed{0} \end{vmatrix}. \quad (18)$$

Next, from (55), we have

$$\begin{aligned}
(g_{l-1}^{-1})_{\bar{z}} = & \begin{vmatrix} \Delta_{0,\bar{z}} & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_{1,\bar{z}} & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & & \vdots \\ \Delta_{l-2,\bar{z}} & \Delta_{l-3} & \cdots & \Delta_{-1} \\ \boxed{\Delta_{l-1,\bar{z}}} & \Delta_{l-2} & \cdots & \Delta_0 \end{vmatrix} \\
& + \sum_{k=1}^{l-1} \begin{vmatrix} \Delta_{-k,\bar{z}} & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_{1-k,\bar{z}} & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & & \vdots \\ \Delta_{l-2-k,\bar{z}} & \Delta_{l-3} & \cdots & \Delta_{-1} \\ \boxed{\Delta_{l-1-k,\bar{z}}} & \Delta_{l-2} & \cdots & \Delta_0 \end{vmatrix} \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-k} & \cdots & \Delta_{1-l} \\ \vdots & \vdots & & \vdots & & \vdots \\ \Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_{l-2-k} & \cdots & \Delta_{-1} \\ \boxed{0} & 0 & \cdots & 1 & \cdots & 0 \end{vmatrix}.
\end{aligned}$$

The effect of the left and right factors on this expression is to move expansion points as specified in (54), obtaining

$$\begin{aligned}
f_{l-1}^{-1} g_{l-1,\bar{z}} b_{l-1}^{-1} = & - \begin{vmatrix} \Delta_{-1} & \cdots & \Delta_{1-l} & \boxed{\Delta_{1-l,\bar{z}}} \\ \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{2-l,\bar{z}} \\ \vdots & & \vdots & \vdots \\ \Delta_{l-3} & \cdots & \Delta_{-1} & \Delta_{-1,\bar{z}} \\ \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{0,\bar{z}} \end{vmatrix} \\
& - \sum_{k=0}^{l-2} \begin{vmatrix} \Delta_{-1} & \cdots & \Delta_{1-l} & \boxed{\Delta_{-k,\bar{z}}} \\ \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-k,\bar{z}} \\ \vdots & & \vdots & \vdots \\ \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{l-1-k,\bar{z}} \end{vmatrix} \begin{vmatrix} 0 & \cdots & 1 & \cdots & 0 & \boxed{0} \\ \Delta_0 & \cdots & \Delta_{-k} & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & & \vdots & & \vdots & \vdots \\ \Delta_{l-2} & \cdots & \Delta_{l-2-k} & \cdots & \Delta_0 & \Delta_{-1} \end{vmatrix}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
e'_{l,w} = & \begin{vmatrix} \Delta_{-1} & \cdots & \Delta_{1-l} & \boxed{\Delta_{-l,w}} \\ \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l,w} \\ \vdots & & \vdots & \vdots \\ \Delta_{l-3} & \cdots & \Delta_{-1} & \Delta_{-2,w} \\ \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1,w} \end{vmatrix} \\
& + \sum_{k=0}^{l-2} \begin{vmatrix} \Delta_{-1} & \cdots & \Delta_{1-l} & \boxed{\Delta_{-k-1,w}} \\ \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{-k,w} \\ \vdots & & \vdots & \vdots \\ \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{l-2-k,w} \end{vmatrix} \begin{vmatrix} 0 & \cdots & 1 & \cdots & 0 & \boxed{0} \\ \Delta_0 & \cdots & \Delta_{-k} & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & & \vdots & & \vdots & \vdots \\ \Delta_{l-2} & \cdots & \Delta_{l-2-k} & \cdots & \Delta_0 & \Delta_{-1} \end{vmatrix},
\end{aligned}$$

and then the result follows immediately from $\Delta_{i,w} = \Delta_{i+1,\bar{z}}$ in (15).

We can find that the proof of these results relies on using quasideterminant identities alone. Thus we can conclude that *noncommutative Bäcklund transformations are identities of quasideterminants*.

We can also present a compact form of the whole of Yang's J -matrix in terms of a single quasideterminant expanded by a 2×2 submatrix:

$$\begin{vmatrix} a & b & c \\ d & \boxed{e} & \boxed{f} \\ g & \boxed{h} & \boxed{i} \end{vmatrix} := \left[\begin{vmatrix} a & b \\ d & \boxed{e} \end{vmatrix} \begin{vmatrix} a & c \\ d & \boxed{f} \end{vmatrix} \right].$$

The solutions for the J -matrix can be presented as follows:

- noncommutative Atiyah-Ward ansatz solutions R_l

$$J_l = \begin{vmatrix} \boxed{0} & -1 & 0 & \cdots & 0 & \boxed{0} \\ 1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\ \boxed{0} & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \boxed{\Delta_0} \end{vmatrix}, \quad J_l^{-1} = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & \boxed{0} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 & 1 \\ \boxed{0} & 0 & \cdots & 0 & -1 & \boxed{0} \end{vmatrix}.$$

- noncommutative Atiyah-Ward ansatz solutions R'_l

$$J'_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} & -1 \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \boxed{\Delta_0} & \boxed{0} \\ 1 & 0 & \cdots & 0 & \boxed{0} & \boxed{0} \end{vmatrix}, \quad J'^{-1}_l = \begin{vmatrix} \boxed{0} & \boxed{0} & 0 & \cdots & 0 & 1 \\ \boxed{0} & \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\ 0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\ -1 & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 \end{vmatrix}.$$

Because J is gauge invariant, this shows that the present Bäcklund transformation is not just a gauge transformation but a nontrivial transformation.

The proof of these representations is given by using the noncommutative Jacobi identity, homological relations and the inversion formula for J :

$$J^{-1} = \begin{bmatrix} f^{-1} & f^{-1}g \\ -ef^{-1} & b - ef^{-1}g \end{bmatrix}, \quad (19)$$

or simply using the formula (56). (For a detailed proof, see Appendix A in Gilson, Hamanaka and Nimmo (2009).)

(c) *Some Explicit Examples*

By solving the noncommutative linear equation $(\partial_z \partial_{\tilde{z}} - \partial_w \partial_{\tilde{w}}) \Delta_0 = 0$ for the seed solution of the Bäcklund transformations, we can obtain exact solutions explicitly.

For example, in Euclidean space, the noncommutative linear equation is just the 4-dimensional noncommutative Laplace equation whose solutions include a noncommutative version of the fundamental solution: $\Delta_0 = 1 + \sum_{p=1}^k (a_p / (z\tilde{z} - w\tilde{w}))$ (a_p are constants), which leads to noncommutative instanton solutions whose instanton number is k (Correa, Lozano, Moreno and Schaposnik (2001); Lechtenfeld and Popov (2002); Nekrasov and Schwarz (1998)). The Bäcklund transformations do not increase the instanton number.

There is also a simple new solution:

$$\Delta_0 = c \exp(az + b\tilde{z} + aw + b\tilde{w}), \quad (20)$$

where a, b and c are constants. This leads to a noncommutative version of nonlinear plane wave solutions (de Vega (1988)). These solutions behave as standard solitons in lower-dimension and do not decay at infinity, which implies that this gives an infinite value for the Yang-Mills action. By following the analysis used in Hamanaka (2007) for the noncommutative KP equation, the asymptotic behaviour of these solutions can be shown to be the same as the corresponding commutative ones.

Other solutions are also easily obtained and a more detailed discussion on this topic will be reported elsewhere.

4. Twistor descriptions of the noncommutative ASDYM equations

In this section, we explain the origin of the Bäcklund transformations for the noncommutative ASDYM equations and noncommutative Atiyah-Ward ansatz solutions from the geometrical viewpoint of noncommutative twistor theory. Here we just need a one-to-one correspondence between a solution of the noncommutative ASDYM equations and a patching matrix $P = P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)$ of a noncommutative holomorphic vector bundle on a noncommutative 3-dimensional projective space, which is called the noncommutative Penrose-Ward correspondence. This correspondence is established in the Moyal-deformed case by Brain (2005); Brain and Majid (2008); Kapustin, Kuznetsov and Orlov (2001); Lechtenfeld and Popov (2002); Takasaki (2001) and here we apply their formal procedure to general noncommutative situations. Such twistor treatments are useful not only for constructing exact solutions but also for checking whether the Bäcklund transformation act on the solution spaces transitively.

In order to review this correspondence briefly, we introduce another linear system defined on another local patch whose (commutative) coordinate is $\tilde{\zeta} = 1/\zeta$,

$$\begin{aligned} (\tilde{\zeta} D_w - D_{\tilde{z}}) \tilde{\psi} &= 0, \\ (\tilde{\zeta} D_z - D_{\tilde{w}}) \tilde{\psi} &= 0. \end{aligned} \quad (21)$$

A nontrivial solution $\tilde{\psi}$ ($N \times N$ matrix) of the linear system (21) is supposed to exist and is not regular at $\tilde{\zeta} = \infty$ (or equivalently $\zeta = 0$) as discussed earlier for ψ .

Any solution of the noncommutative ASDYM equation determines solution ψ and $\tilde{\psi}$ that are unique up to gauge transformation, and then the corresponding patching matrix is given by

$$P(x; \zeta) = \tilde{\psi}^{-1}(x; \zeta) \psi(x; \zeta). \quad (22)$$

Conversely, if a patching matrix $P = P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)$ is factorized as

$$P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{\psi}^{-1}(x; \zeta) \psi(x; \zeta), \quad (23)$$

where ψ and $\tilde{\psi}$ are regular near $\zeta = 0$ and $\zeta = \infty$, respectively, then ψ and $\tilde{\psi}$ are solutions of the linear system (1) for the noncommutative ASDYM equation. Then we can recover the ASDYM gauge fields in terms of h and \tilde{h} by using (7) and the fact that $h(x) = \psi(x, \zeta = 0)$, $\tilde{h}(x) = \tilde{\psi}(x, \zeta = \infty)$. (We can easily understand this by comparing the linear systems (1) and (21) with (6).)

In the commutative case, if P is holomorphic w.r.t. ζ , then the factorization is guaranteed by the Birkhoff factorization theorem. In the case of the Moyal deformation, this is formally proved by Takasaki (2001). Here we will see that under the Atiyah-Ward ansatz for the patching matrix, the factorization problem (the *Riemann-Hilbert problem*) is solved.

In this section, we fix the gauge to be, what we call in this paper, the *Mason-Woodhouse gauge*

$$J = \begin{bmatrix} f - gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{bmatrix} = \begin{bmatrix} 1 & g \\ 0 & b \end{bmatrix}^{-1} \begin{bmatrix} f & 0 \\ e & 1 \end{bmatrix} = \tilde{h}_{\text{MW}}^{-1} h_{\text{MW}}. \quad (24)$$

We note that the gauge transformation $g = \text{diag}(f^{1/2}, b^{1/2})$ connects the Mason-Woodhouse gauge with a noncommutative version of Yang's R -gauge (Yang (1977)):

$$J = \begin{bmatrix} f - gb^{-1}e & -gb^{-1} \\ b^{-1}e & b^{-1} \end{bmatrix} = \begin{bmatrix} f^{-1/2} & f^{-1/2}g \\ 0 & b^{1/2} \end{bmatrix}^{-1} \begin{bmatrix} f^{1/2} & 0 \\ b^{-1/2}e & b^{-1/2} \end{bmatrix} = \tilde{h}_{\text{R}}^{-1} h_{\text{R}}, \quad (25)$$

where the square root is considered as any quantity which satisfies $f^{1/2} f^{1/2} = f$, $f^{-1/2} := (f^{1/2})^{-1}$ and whenever this notation is used, it is assumed to exist.

The wave functions ψ and $\tilde{\psi}$ can be expanded by ζ and $\tilde{\zeta} = 1/\zeta$, respectively:

$$\begin{aligned} \psi &= h + \mathcal{O}(\zeta) = \begin{bmatrix} h_{11} + \sum_{i=1}^{\infty} a_i \zeta^i & h_{12} + \sum_{i=1}^{\infty} b_i \zeta^i \\ h_{21} + \sum_{i=1}^{\infty} c_i \zeta^i & h_{22} + \sum_{i=1}^{\infty} d_i \zeta^i \end{bmatrix}, \\ \tilde{\psi} &= \tilde{h} + \mathcal{O}(\tilde{\zeta}) = \begin{bmatrix} \tilde{h}_{11} + \sum_{i=1}^{\infty} \tilde{a}_i \tilde{\zeta}^i & \tilde{h}_{12} + \sum_{i=1}^{\infty} \tilde{b}_i \tilde{\zeta}^i \\ \tilde{h}_{21} + \sum_{i=1}^{\infty} \tilde{c}_i \tilde{\zeta}^i & \tilde{h}_{22} + \sum_{i=1}^{\infty} \tilde{d}_i \tilde{\zeta}^i \end{bmatrix}. \end{aligned} \quad (26)$$

(a) *Riemann-Hilbert problem for noncommutative Atiyah-Ward Ansatz*

From now on, we restrict ourselves to $G = GL(2)$. In this case, we can take a simple ansatz for the patching matrix P , which is called the Atiyah-Ward ansatz in the commutative case (Atiyah and Ward (1978)). The noncommutative generalization of this ansatz is straightforward and actually leads to a solution of the factorization problem. The l -th order noncommutative Atiyah-Ward ansatz ($l = 0, 1, 2, \dots$) is specified by choosing the patching matrix to be:

$$P_l(x; \zeta) = \begin{bmatrix} 0 & \zeta^{-l} \\ \zeta^l & \Delta(x; \zeta) \end{bmatrix}. \quad (27)$$

(The standard representation of the ansatz is not P_l but $C_0 P_l$. Both representations are essentially the same.) We note that the coordinate dependence of $P_l = P_l(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)$ implies that $(\partial_w - \zeta \partial_{\tilde{z}})\Delta = 0$, $(\partial_z - \zeta \partial_{\tilde{w}})\Delta = 0$. Hence, the Laurent expansion of Δ w.r.t. ζ

$$\Delta(x; \zeta) = \sum_{i=-\infty}^{\infty} \Delta_i(x) \zeta^{-i}, \quad (28)$$

gives rise to the following relationships amongst the coefficients,

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}, \quad (29)$$

which coincide with the recurrence relation (15). We will soon see that the coefficients $\Delta_i(x)$ are the scalar functions in the solutions generated by the Bäcklund transformations in the previous section.

We will now solve the factorization problem $\tilde{\psi} P_l = \psi$ for the noncommutative Atiyah-Ward ansatz. In explicit form this is

$$\begin{bmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} \\ \tilde{\psi}_{21} & \tilde{\psi}_{22} \end{bmatrix} \begin{bmatrix} 0 & \zeta^{-l} \\ \zeta^l & \Delta(x; \zeta) \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}, \quad (30)$$

where ψ_{ij} is the (i, j) -th element of ψ , and so,

$$\tilde{\psi}_{12} \zeta^l = \psi_{11}, \quad \tilde{\psi}_{22} \zeta^l = \psi_{21}, \quad (31)$$

$$\tilde{\psi}_{11} \zeta^{-l} + \tilde{\psi}_{12} \Delta = \psi_{12}, \quad \tilde{\psi}_{21} \zeta^{-l} + \tilde{\psi}_{22} \Delta = \psi_{22}. \quad (32)$$

From (26) and (31), we find that some entries in ψ and $\tilde{\psi}$ are polynomials w.r.t. ζ and $\tilde{\zeta} = \zeta^{-1}$:

$$\begin{aligned} \psi_{11} &= h_{11} + a_1 \zeta + a_2 \zeta^2 + \dots + a_{l-1} \zeta^{l-1} + \tilde{h}_{12} \zeta^l, \\ \psi_{21} &= h_{21} + c_1 \zeta + c_2 \zeta^2 + \dots + c_{l-1} \zeta^{l-1} + \tilde{h}_{22} \zeta^l, \\ \tilde{\psi}_{12} &= \tilde{h}_{12} + a_{l-1} \zeta^{-1} + a_{l-2} \zeta^{-2} + \dots + a_1 \zeta^{1-l} + h_{11} \zeta^{-l}, \\ \tilde{\psi}_{22} &= \tilde{h}_{22} + c_{l-1} \zeta^{-1} + c_{l-2} \zeta^{-2} + \dots + c_1 \zeta^{1-l} + h_{21} \zeta^{-l}. \end{aligned} \quad (33)$$

By substituting these formulae into (32), we get the following sets of equations for h and \tilde{h} in the coefficients of $\zeta^0, \zeta^{-1}, \dots, \zeta^{-l}$:

$$\begin{aligned} (h_{11}, a_1, \dots, a_{l-1}, \tilde{h}_{12}) D_{l+1} &= (-\tilde{h}_{11}, 0, \dots, 0, h_{12}), \\ (h_{21}, c_1, \dots, c_{l-1}, \tilde{h}_{22}) D_{l+1} &= (-\tilde{h}_{21}, 0, \dots, 0, h_{22}), \end{aligned} \quad (34)$$

where

$$D_l := \begin{bmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \end{bmatrix}. \quad (35)$$

These noncommutative linear equations can be solved in terms of quasideterminants (cf. (49)) as

$$\begin{aligned} h_{11} &= h_{12} |D_{l+1}|_{1,l+1}^{-1} - \tilde{h}_{11} |D_{l+1}|_{1,1}^{-1}, \\ h_{21} &= h_{22} |D_{l+1}|_{1,l+1}^{-1} - \tilde{h}_{21} |D_{l+1}|_{1,1}^{-1}, \\ \tilde{h}_{12} &= h_{12} |D_{l+1}|_{l+1,l+1}^{-1} - \tilde{h}_{11} |D_{l+1}|_{l+1,1}^{-1}, \\ \tilde{h}_{22} &= h_{22} |D_{l+1}|_{l+1,l+1}^{-1} - \tilde{h}_{21} |D_{l+1}|_{l+1,1}^{-1}. \end{aligned} \quad (36)$$

In (36) there are four equations but eight unknowns and so in order to solve them it is necessary to impose four conditions corresponding to a choice of gauge. In particular, the Mason-Woodhouse gauge ($h_{12} = \tilde{h}_{21} = 0, \tilde{h}_{11} = h_{22} = 1$) leads to simple representations:

$$\begin{aligned} h_{11} &= - \left| \begin{array}{cccc} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{array} \right|^{-1}, & h_{21} &= - \left| \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \boxed{\Delta_{-l}} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{array} \right|^{-1}, \\ \tilde{h}_{12} &= - \left| \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\Delta_l} & \Delta_{l-2} & \cdots & \Delta_0 \end{array} \right|^{-1}, & \tilde{h}_{22} &= - \left| \begin{array}{cccc} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \boxed{\Delta_0} \end{array} \right|^{-1}, \end{aligned} \quad (37)$$

which coincides exactly with the solutions R_l generated by the Bäcklund transformation in the previous section except for signs in f_l and g_l . (The mismatch of the signs is not essential because it can be absorbed into the reflection symmetry $f \mapsto -f, g \mapsto -g$ of the noncommutative Yang equation (12).) That is why we call them the noncommutative Atiyah-Ward ansatz solutions. The class of solutions R'_l is also obtained in the same way by starting with the Atiyah-Ward ansatz $C_0^{-1} P_l C_0$.

(b) *Origin of the noncommutative Corrigan-Fairlie-Yates-Goddard transformation*

Finally let us discuss the origin of the noncommutative Corrigan-Fairlie-Yates-Goddard transformation, constructed from the β -transformation and the γ_0 -transformation and give a generalization of it. Such geometrical understanding is useful when discussing whether the group action of the Bäcklund transformations is transitive and hence to find the symmetry of the noncommutative ASDYM equation. The present results are essentially due to Mason, Chakravarty and Newman (1988), Mason and Woodhouse (1996). These transformations can be viewed as adjoint actions of the patching matrix P :

$$\beta : P \mapsto P^{\text{new}} = B^{-1}PB, \quad \gamma_0 : P \mapsto P^{\text{new}} = C_0^{-1}PC_0, \quad (38)$$

where

$$B = \begin{bmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (39)$$

It is obvious that $\beta \circ \beta = id, \gamma_0 \circ \gamma_0 = id$.

The composition of these transformations actually maps the l -th Atiyah-Ward ansatz to the $(l+1)$ -th one:

$$P_l \mapsto C_0^{-1}B^{-1} \begin{bmatrix} 0 & \zeta^{-l} \\ \zeta^l & \Delta \end{bmatrix} BC_0 = \begin{bmatrix} 0 & \zeta^{-(l+1)} \\ \zeta^{l+1} & \Delta \end{bmatrix} = P_{l+1}. \quad (40)$$

The action of C_0 leads to $h \mapsto hC_0, \tilde{h} \mapsto \tilde{h}C_0$ and hence to the γ_0 -transformation, and the action of B is defined at the level of ψ and $\tilde{\psi}$ as follows:

$$\psi^{\text{new}} = g^{-1}\psi B, \quad \tilde{\psi}^{\text{new}} = g^{-1}\tilde{\psi} B, \quad (41)$$

where

$$g^{-1} = \begin{bmatrix} 0 & \zeta b^{-1} \\ f^{-1} & 0 \end{bmatrix}. \quad (42)$$

The gauge transformation g is needed to maintain the regularity of ψ and $\tilde{\psi}$, w.r.t ζ and $\tilde{\zeta}$ respectively, in the factorization of P .

The explicit calculation gives

$$\psi^{\text{new}} = \begin{bmatrix} b^{-1}\psi_{22} & \zeta b^{-1}\psi_{21} \\ \zeta^{-1}f^{-1}\psi_{12} & f^{-1}\psi_{11} \end{bmatrix}. \quad (43)$$

In the $\zeta \rightarrow 0$ limit, this reduces to

$$h^{\text{new}} = \begin{bmatrix} f^{\text{new}} & 0 \\ e^{\text{new}} & 1 \end{bmatrix} = \begin{bmatrix} b^{-1} & 0 \\ f^{-1}k_{12} & 1 \end{bmatrix}, \quad (44)$$

where $\psi = h + k\zeta + \mathcal{O}(\zeta^2)$.

Here we note that the linear system (1) can be represented in terms of b, f, e, g as

$$\begin{aligned} L\psi &= (\partial_w - \zeta \partial_{\bar{z}})\psi + \begin{bmatrix} -f_w f^{-1} & \zeta g_{\bar{z}} b^{-1} \\ -e_w f^{-1} & \zeta b_{\bar{z}} b^{-1} \end{bmatrix} \psi = 0, \\ M\psi &= (\partial_z - \zeta \partial_{\bar{w}})\psi + \begin{bmatrix} -f_z f^{-1} & \zeta g_{\bar{w}} b^{-1} \\ -e_z f^{-1} & \zeta b_{\bar{w}} b^{-1} \end{bmatrix} \psi = 0. \end{aligned} \quad (45)$$

By considering the first order term of ζ in the (1,2) component of the first equation, we find that

$$\partial_w(f^{-1}k_{12}) = -f^{-1}g_{\bar{z}}b^{-1}. \quad (46)$$

Hence from the (1,1) and (2,1) components of (44), we have

$$f^{\text{new}} = b^{-1}, \quad \partial_w e^{\text{new}} = \partial_w(f^{-1}k_{12}) = -f^{-1}g_{\bar{z}}b^{-1}, \quad (47)$$

which are just parts of the β -transformation (13). In similar way, we can get the other ones. Therefore the β -transformation (13) can be interpreted as the transformation of the patching matrix $P \mapsto B^{-1}PB$ together with the gauge transformation g . The results presented in the section and the previous one lead to a simpler proof of the results in Section 3.

We note that the γ_0 -transformation can be generalized to the following transformation (the γ -transformation):

$$\gamma : P \mapsto P^{\text{new}} = C^{-1}PC, \quad (48)$$

where C is an arbitrary constant matrix. The actions of β - and γ - transformations generate the action of the loop group $LGL(2)$ on P by conjugation. Therefore the symmetry group of the noncommutative ASDYM equation includes the loop group $LGL(2)$ as a subgroup.

5. Conclusion and Discussion

In this paper, we have presented Bäcklund transformations for the noncommutative ASDYM equation with $G = GL(2)$ and constructed from a simple seed solution a series of exact noncommutative Atiyah-Ward ansatz solutions expressed explicitly in terms of quasideterminants. We have found that the Bäcklund transformations generate a wide class of new solutions. We have also given the origin of the Bäcklund transformation and the generated solutions in the framework of noncommutative twistor theory and generalized them.

The present results could be taken as the starting point to reveal an infinite-dimensional symmetry of the noncommutative ASDYM equation in terms of some infinite-dimensional algebra. We have to prove that the Atiyah-Ward ansatz covers all solutions of noncommutative ASDYM equation and generalize the Bäcklund transformations β and γ so that they should act on the solution space transitively.

Furthermore investigation of the noncommutative extension of a bilinear form approach to the ASDYM equation (Gilson, Nimmo and Ohta (1998); Sasa, Ohta and Matsuikidaira (1998); Wang and Wadati (2004)) would be beneficial because many aspects in these paper are close to ours. The relationship with noncommutative Darboux and noncommutative binary Darboux transformations (Salaam, Hassan and Siddiq (2007)) is also interesting.

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A. Brief review of quasideterminants

In this section, we give a brief introduction to quasideterminants, introduced by Gelfand and Retakh (2001), in which a few of the key properties which play important roles in Section 3 are described. More detailed discussion is seen in the survey (Gelfand, Gelfand, Retakh and Wilson (2005)).

Quasideterminants are defined in terms of inverse matrices and we suppose the existence of all matrix inverses referred to. Let $A = (a_{ij})$ be an $n \times n$ matrix and $B = (b_{ij})$ be the inverse matrix of A , that is, $AB = BA = 1$. Here the matrix entries belong to a noncommutative ring. Quasideterminants of A are defined formally as the inverses of the entries in B :

$$|A|_{ij} := b_{ji}^{-1}. \quad (49)$$

In the case that variables commute, this is reduced to

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}, \quad (50)$$

where A^{ij} is the matrix obtained from A by deleting the i -th row and the j -th column.

We can also write down a more explicit definition of quasideterminants. In order to see this, let us recall the following formula for the inverse a square 2×2 block square matrix:

$$\begin{bmatrix} A & B \\ C & d \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix},$$

where A is a square matrix, d is a single element and B and C are column and row vectors of appropriate length and $S = d - CA^{-1}B$ is called a *Schur complement*. In fact this formula is valid for A , B , C and d in any ring not just for matrices. Thus the

quasideterminant associated with the bottom right element is simply S . By choosing an appropriate partitioning, any entry in the inverse of a square matrix can be expressed as the inverse of a Schur complement and hence quasideterminants can also be defined recursively by:

$$|A|_{ij} = a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'}((A^{ij})^{-1})_{i'j'}a_{j'j} = a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'}(|A^{ij}|_{j'i'})^{-1}a_{j'j}. \quad (51)$$

It is sometimes convenient to use the following alternative notation in which a box is drawn about the corresponding entry in the matrix:

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & \boxed{a_{ij}} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}. \quad (52)$$

Quasideterminants have various interesting properties similar to those of determinants. Among them, the following ones play important roles in this paper. In the block matrices given in these results, lower case letters denote single entries and upper case letters denote matrices of compatible dimensions so that the overall matrix is square.

- noncommutative Jacobi identity

A simple and useful special case of the noncommutative Sylvester's Theorem (Gelfand and Retakh (1991)) is

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}. \quad (53)$$

- Homological relations (Gelfand and Retakh (1991))

$$\begin{aligned} \begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} &= \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix}, \\ \begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & i \end{vmatrix} &= \begin{vmatrix} A & B & 0 \\ D & f & \boxed{0} \\ E & h & 1 \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \end{aligned} \quad (54)$$

- A derivative formula for quasideterminants (Gilson and Nimmo (2009))

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B' \\ C & \boxed{d'} \end{vmatrix} + \sum_{k=1}^n \begin{vmatrix} A & (A_k)' \\ C & \boxed{(C_k)'} \end{vmatrix} \begin{vmatrix} A & B \\ e_k^t & \boxed{0} \end{vmatrix}, \quad (55)$$

where A_k is the k th column of a matrix A and e_k is the column n -vector (δ_{ik}) (i.e. 1 in the k th row and 0 elsewhere).

- A special formula of inverse of a quasideterminant (Gilson, Hamanaka and Nimmo (2007))

$$\begin{vmatrix} a & B & c & \alpha \\ D & E & F & 0 \\ g & H & \boxed{i} & \boxed{0} \\ \beta & 0 & \boxed{0} & \boxed{0} \end{vmatrix}^{-1} = \begin{vmatrix} \boxed{0} & \boxed{0} & 0 & \gamma \\ \boxed{0} & \boxed{a} & B & c \\ 0 & D & E & F \\ \delta & g & H & i \end{vmatrix}, \quad (56)$$

with $\alpha\beta = \gamma\delta = -1$, $\alpha + \gamma = 0$, where lower case letters denote single entries, upper case letters denote matrices of compatible dimensions and Greek letters are scalars (i.e. commute with everything).

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